The slower lane paradox, or: you're not paranoid, the universe really is out to get you

Gaurav Shah
gaurav.shah.contact@gmail.com

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1 A simple description

If you are in one of two identical lanes, then for the majority of the time you'll be in the slower lane. This fact does not spring from the psychology of human perception, but from mathematics.[1]

This violates our sense of symmetry, and so can be considered a paradox. The explanation of this paradox is not terribly sophisticated mathematically, although there are some subtleties to do with probability. However, to the best of my knowledge, it hasn't been described elsewhere; at least, it isn't familiar to most people. Since this situation comes up all so often, it seems it should be better known.

And it's a nice little exercise in simple probability, too, so we'll work it out in detail. Even if the result is obvious, there are some interesting aspects.

(This of course extends to the case of more than two lanes quite naturally.)

2 Setup

We'll take the simplest possible scenario.[2] Consider a highway with only two lanes, lane A and lane B, going from StartCity to EndCity.

- We assume the traffic can only be at one of two speeds, v_1 and v_2. Assume $v_2 > v_1$.
- Lane A and lane B are *statistically identical*. They are both equally likely to be at speed v_1, and equally likely to be at speed v_2. Assume that each is at

[1] The psychological aspect will enthusiastically add to our travails, of course.

[2] Although we'll go into arguably unnecessary detail and precision specifying the scenario.

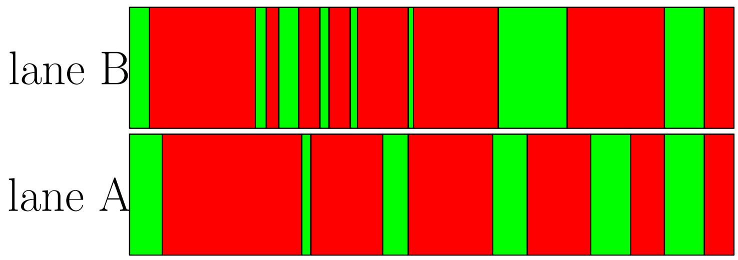

Figure 1: red $= v1$, green $= v2$. Total length of red is equal in both lanes; similarly for green.

speed v_1 for a total distance of s_1, and at speed v_2 for a total distance of s_2 of the journey. These total distances s_1 and s_2 are split up randomly into segments along the path, separately and independently for each lane.[3]

- There is a reasonably continuous stream of traffic in each lane. In particular, if we point to any particular part of the road, for either lane, you can tell what the speed of the lane at that point is.
- A car, driven by Alice, starts out at $t = 0$ from StartCity in lane A.

The paradox: Alice looks out her window, noticing whether the traffic in lane B is faster than her, or slower. Since each lane is statistically identical, the naive assumption would be that she is faster than the other lane for the same amount of time as she is slower than the other lane.

But in truth, the probability of Alice noticing that the traffic in the other lane is moving *faster* than she is, is **greater than** the probability of seeing that the traffic in the other lane is moving *slower* than she is. In other words, the other lane is faster for most of the time.

3 Proofs

We'll show this in two ways. The first method is more algebraic, which will bring out some of the subtleties more explicitly. The other one is simpler and more

[3]The exact nature of this "randomness" (such as the spectrum of sizes of each segment) is something we will not delve into, as it's not important for this discussion.

conceptual, but I think it glosses over some of the points unless you closely ponder the assumptions being made.

3.1 Algebraically, and excruciatingly detailed

Step 1 The probability of Alice being at speed $v1$ or $v2$ is different, depending on whether you ask the probability at a given *position*, or at a given *time*.

If she is travelling at v_1 for a distance of d_1, and at v_2 for a distance of d_2, then the probability of travelling at v_1 at a given position in the journey is $p_1^{(d)} := \frac{d_1}{d_1+d_2}$.

The probability of travelling at v_1 at a given time in the journey, on the other hand, is $p_1^{(t)} := \frac{d_1/v_1}{d_1/v_1+d_2/v_2}$.

Note that because the lanes are statistically identical, the values of these probabilities are the same for Lane A and for Lane B.

Step 2 Let the probability that, at a given time t, Alice is moving faster than the traffic in Lane B next to her be denoted $p_>^{(t)}$, and that she's slower than the traffic in Lane B as $p_<^{(t)}$.

She will be moving *faster* than the traffic in Lane B if

1. At time t, she is travelling at speed v_2 (which we denote p_1), *and*
2. Lane B next to her is travelling at speed v_1 (which we denote p_2).

Since the lanes are independent, $p_>^{(t)} = p_1 \times p_2$.

Here is the crux of the argument.

Since the first event, Alice travelling at speed v_2, is at a specific *time* t, the correct probability is the probability that she's travelling v_2 at a given time, which we've already called $p_2^{(t)}$.

But the second event, that the traffic in Lane B is at v_1, is at a specific *location* – namely, next to Alice. So the probability of it is the probability of traffic travelling at v_1 at a given position, which we know as $p_1^{(d)}$.

$$\begin{aligned} p_>^{(t)} &= p_1 \times p_2 \\ &= p_2^{(t)} \times p_1^{(d)} \\ &= \frac{d_2/v_2}{d_1/v_1 + d_2/v_2} \times \frac{d_1}{d_1 + d_2} \end{aligned}$$

Using the same logic,

$$p_{<}^{(t)} = p_1^{(t)} \times p_2^{(d)} = \frac{d_1/v_1}{d_1/v_1 + d_2/v_2} \times \frac{d_2}{d_1 + d_2}$$

So

$$p_{>}^{(t)}/p_{<}^{(t)} = v_1/v_2$$

Since $v_1 < v_2$, $p_{>}^{(t)}/p_{<}^{(t)} < 1$.

In other words, **Alice spends more time going slower than the traffic in the other lane than she does going faster than it**.

3.2 Proof 2, much simpler

The total distance along the path where Lane A is faster than Lane B is equal to the total distance where Lane A is slower than Lane B, since they're statistically identical.

But Alice spends less time in the part where she's going faster, since she's at v_2 in that part, and v_1 in the other part.